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ON COMBINATION RESONANCES IN HYDRODYNAMICS

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The problem of small amplitude resonance oscillations are considered in a nonlinear system close to its stability limit. Solution is sought in the form of series in terms of natural oscillations and of inducing force. The obtained equation which defines the dependence of natural oscillation amplitude on time is used for analyzing the evolution of steady motions with varying parameters. Subharmonic oscillations of polytropic gas, induced by a piston in a long pipe ending in a chamber is considered as an example.

1. Combination resonances may occur in systems that are defined by equations of the hydrodynamic type, when these are subjected to periodic forces. In self-excited systems modes of beat and forced oscillations are possible; an example of this is provided by the combination resonances in gas discharge [1].

The principal resonance was considered in [2]. Combination resonances require special investigation, since the character of oscillations and their amplitude substantially depend on the kind of resonance.

As in [2] we consider the boundary value problem

$$\frac{\partial X}{\partial t} + L_1X + L_2X^2 + \ldots = \varepsilon EC + \text{c.c.}, \quad UX = 0 \quad (E = e^{i\omega t}) \qquad (1.1)$$

where X is the vector of small deviations of variables from their equilibrium values, and c. c. denotes an expression that is a complex conjugate of the preceding one.

Real coefficients L and matrix U depend in the boundary condition on parameters λ and x-coordinates, and are polynomials in $D = \partial / \partial x$. The region of varia- of x is assumed to be bounded. The periodic perturbation of the form $C = C(x)$ is proportional to the small amplitude ε. It is assumed that the problem

$$p_0X_1 + L_1X_1 = 0, \quad UX_1 = 0 \qquad (1.2)$$

has a simple eigenvalue $p_0 = \gamma_0 + i\Omega_0$ (below it is called critical) which for $\lambda = \lambda_*$ becomes pure imaginary ($p_0 = i\Omega_*$, $\Omega_* > 0$); the increments $\gamma_0(\lambda_*)$ of other eigenvalues are negative and are not small.

When $\varepsilon \neq 0$ a combination resonance occurs, if for the irreducible fraction $\zeta = n / m$, where m and n are positive integers, the frequency difference $p = p_0 - i\omega\zeta$ is small in comparison with the quantities $\omega(1, \zeta, 1 - \zeta)$. The small parameters γ_0, $\Omega_0 - \omega\zeta$ and ε are independent.

The solution of problem (1.1) is sought in the form of series in powers of the small quantities QE^ζ, εE and their complex conjugate

$$X = (QE^\zeta X_1 + \varepsilon EX_2) + \text{c.c.} + \ldots \qquad (1.3)$$

Coefficients of the series depend only on x and X_1 is the eigenfunction of problem (1.2), which corresponds to the critical p_0.

The equation for the amplitude of oscillations $Q(t)$ is also sought in the form of series in Q, ε and their complex conjugate

$$\frac{dQ}{dt} = Q(p + p_3\varepsilon\bar{\varepsilon} + p_2Q\bar{Q} + \ldots) + \bar{Q}^{m-1}\varepsilon^n(p_1 + \ldots) + \ldots \tag{1.4}$$

where series in powers of $|\varepsilon|^2$ and $|Q|^2$ appear in parentheses, and the series multiplied by $\bar{Q}^{ms-1}\varepsilon^{ns}$ when $s > 1$, are omitted.

Coefficients in (1.3) and (1.4) are determined by nonhomogeneous linear problems that obtain after the substitution of (1.3) and (1.4) into (1.1), and by equating terms proportional to $Q^a\bar{Q}^b\varepsilon^c\bar{\varepsilon}^dE^{\zeta\nu}$, where $\nu = a - b + (c - d)/\zeta$. It is sufficient to consider problems for $\nu \geqslant 0$. Solutions of such problems are finite when $\nu \neq 1$, hence the related coefficient in (1.4) is zero. When $\nu = 1$ the solutions are finite for $p \to 0$, only if

$$\langle\Psi\cdot Z\rangle \equiv \int(\Psi\cdot\bar{Z})\,dx = 0 \tag{1.5}$$

where Ψ is the free term of the nonhomogeneous problem, Z is the eigenfunction of the problem conjugate of problem (1.2) with the critical value p_0; integration is carried out over the region of variation of x. The related coefficient of series (1.4), which is linear in Ψ is determined by (1.5).

The equality $\nu = 1$ is satisfied when $b = a - 1 + ms$, $c = d + ns$ and $s \geqslant 0$. When $s = 0$ the last relationships are independent of numbers m and n which determine the type of resonance. The corresponding terms in (1.4) define nonresonance effects, in particular, the nonsynchronous effect of frequency difference in $(b = 0)$ change. The changed frequency difference is $p_+(\varepsilon) = + d\ p_3\varepsilon\bar{\varepsilon} + \ldots$.

The nonsingular case, when in the approximate equation (1.4) the quantities $p_n = \gamma_n + i\Omega_n$ $(n = 1, 3)$ and γ_2 are not small for $\lambda = \lambda_*$ and $p = 0$ is considered below (some particular cases are considered in Sect. 4). In practice it may be more convenient to determine coefficient in (1.3) and (1.4) in the form of series in $\lambda - \lambda_*$ and $\omega - \Omega_*$.

To determine p_1 it is necessary to take into account in (1.1) all terms of the series with powers $\leqslant N = n + m - 1$; which means that at high resonance orders N resonance effects are negligible. This statement can be substantiated and refined with the use of the following estimates.

When $p_+ = 0$ and $m \leqslant 3$ for the steady nonzero amplitude from (1.4) we obtain $Q \sim \varepsilon^\chi$ where $\chi = n/(4 - m)$. The effects of frequency difference become equal to those of resonance when $p_+ \sim Q^2$; hence for the resonance region (region of forced oscillations) we have $\delta\omega \sim Q^2$. That region and the amplitude decreases with increasing n and m.

For $N = 1$ the first term in (1.3) is the greatest and the nonsynchronous effect is negligible [2].

For $N = 2$ the first term in (1.3) is the greatest in the resonance region. The nonsynchronous effect is negligible in the determination of steady state solution, but has a considerable effect on its stability outside of the resonance region (it is shown below that outside that region $Q \neq 0$ only when $m = 1$).

For $N = 3$ the two terms in (1.3) are of the same order in the resonance region and the nonsynchronous effect is appreciable. For $N > 3$ the second term in (1.3) is the greatest.

When $m > 3$ there is only a trivial steady state solution $Q = 0$ (small for small ε. This means that the region of forced oscillations is absent. Periodic oscillations of small amplitude can only have frequency ω; and they are stable if $\gamma_+ < 0$. The limit cycle with the square of amplitude $|Q|^2 \approx -\gamma_+ / \gamma_2$ and frequency $\Omega = \Omega_+ + |Q|^2\Omega_2$ corresponds to the beat mode (the effect of resonance terms on the cycle index may be determined by the conventional perturbation theory). The cycle is stable when $\gamma_+ > 0$ and $\gamma_2 < 0$ while for $\gamma_+ \geqslant 0$ and $\gamma_2 > 0$ small oscillations are not possible in the system.

The indicated possibilities completely determine the behavior of the system with parameter variation when $m > 3$.

2. To investigate resonance when $m \leqslant 3$ it is convenient to represent (1.4) in the form

$$dR/d\tau = (1 + i\eta) \{R [\alpha (1 - i\sigma) - R\bar{R}] + F\bar{R}^{m-1} + \ldots\} \equiv H \tag{2.1}$$

$$R = (Q / \mu) \exp [i (\varphi_2 - \varphi_1) / m], \quad \tau = t \,|p_2|\, \mu^2 \cos \varphi_2$$

$$\eta = \operatorname{tg} \varphi_2, \quad \sigma = \operatorname{tg} \psi, \quad \psi = \varphi_2 - \varphi, \quad \mu = |(p_+ / p_2) \cos \psi|^{1/2} > 0$$

$$\varepsilon^n p_1 = |\varepsilon^n p_1| \exp i\varphi_1, \quad p_2 = -|p_2| \exp i\varphi_2, \quad p_+ = |p_+| e^{i\varphi}$$

$$F = |\varepsilon^n p_1 / p_2| / \mu^{4-m}, \quad \alpha = (\cos \psi) / |\cos \psi| = \pm 1$$

Equation (2.1) was considered in [3] for $\eta = 0$ and for $m = 1$ it was investigated in [2]. A qualitative investigation of (2.1) is carried out below for $m = 2, 3$ First, the case of $\gamma_2 < 0$ and $\eta \geqslant 0$ is considered.

Structure of the phase plan of Eq.(2.1) is periodic with respect to $\arg R$ of period $2\pi / m$, which implies that the number of singular points with the same $\rho = R\bar{R} \neq 0$ is equal m. For these points from (2.1) we obtain

$$(\alpha - \rho)^2 + \sigma^2 = f\rho^{m-2}, \quad f = F^2 \tag{2.2}$$

It is convenient to consider for $\rho \geqslant 0$ instead of the two equations (2.2) only the one with $\alpha = 1$ in the region $(-\infty < \rho < \infty)$ and use negative ρ for $\alpha = -1$. The amplitude curves are circles $F = \text{const}$ when $m = 2$ and $f = \text{const}$ when $m = 3$ (Fig. 2); curves on which the circles have vertical tangents are shown in Figs. 1 and 2 by dash lines.

Small deviations from equilibrium are proportional to $\varkappa t$, where

$$\varkappa = a \pm (a^2 - b)^{1/2}, \quad a \equiv \operatorname{Re} (\partial H / \partial R) = \alpha (1 + \sigma\eta - 2\rho) \tag{2.3}$$

$$b \equiv |\partial H / \partial R|^2 - |\partial H / \partial \bar{R}|^2 = m (1 + \eta^2) [(2 - m) \times (1 + \sigma^2) - 2\rho (3 - m) + (4 - m) \rho^2]$$

Saddle points lie in region $h < 0$ and along the dash lines where $b = 0$. lie saddle-nodes. Complex focal points lie on straight lines $a = 0$ which intersect axis σ at point $\sigma_1 = -1/\eta$; it can be shown with the use of results obtained in [4] that these focal points are unstable quantities of first multiplicity. Simple focal and nodal points are unstable when in Figs. 1 and 2 they lie between the straight line $\alpha = 0$ and axis σ in the first and third quadrants formed by the σ axis and the straight line $\sigma = \sigma_1$. Along that straight $\gamma_+ = 0$, and in the second and fourth quadrants $\gamma_+ < 0$; passing from the upper half-plane to the lower at change of parameters is only possible through an infinitely distant point and then to the opposite quadrant.

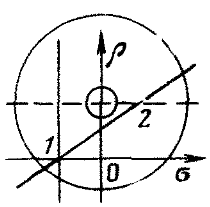

Fig. 1

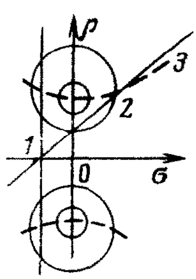

Fig. 2

When $m = 2$ the region of focal points is bounded by a hyperbola with asymptotes $2\rho = 1 - \sigma[1/\eta \pm (1 + 1/\eta^2)^{1/2}]$ and the square of the semiaxis $^1/_8(25 + 16/\eta^2)^{1/2} - {}^3/_8$. When $m = 3$ the region of focal point is bounded by an ellipse (a hyperbola when $\eta > {}^1/_2$) center $(\eta, 2)/(1 - 4\eta^2)$ and the squares of semiaxes $2\eta^2(4 + \eta^2 \pm g)/(1 - 4\eta^2)^{1/2}$ and the direction of the major (real) axis $(7\eta^2 + 2 + g)/(4\eta)$ where $g = [16\eta^2 + (7\eta^2 + 2)^2]^{1/2}$. In the case of equilibrium $R = 0$ we have instead of (2.3) $\alpha_0 = \alpha(1 + \sigma\eta)$,

and $b_0 = (1 + \eta^2)[1 + \sigma^2 - F^2$

For $m = 2$ curves $b_0 = 0$ are shown in Fig. 3 by thin dash lines; saddle points correspond to points of the upper curve, while nodal points correspond to points of the lower curve.

The infinitely distant point is unstable; its index (as well as the combined index of finite points) is unity.

3. Let us consider the subdivision of the parameter plane (Figs. 3 and 4)

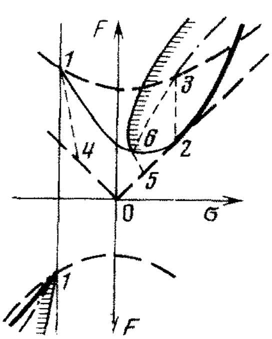

Fig. 3

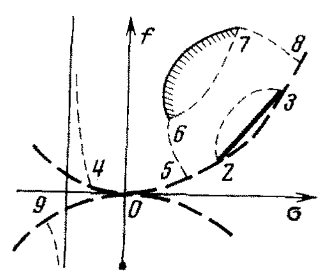

Fig. 4

into regions with constant (or slightly varying) phase structure of the plane of Eq.(2. 1). When that subdivision is known, it is not difficult to determine the behavior of the system at parameter variation.

According to Benickson's criterion there are no limit cycles in the second and fourth quadrants (formed by the σ - axis and the straight line $\sigma = \sigma_1$), since there $a = \alpha(1 + \sigma\eta) - 2R\bar{R} \leqslant 0.$ The integral curves run from infinity to one of the stable singular points.

Limit cycles exist in the first and third quadrants. They comprise one or all singular points, since the combined index of points is unity and the phase plane is periodic with respect to $\arg R$.

Steady cycles comprise unsteady points and point $R = 0.$ The steady cycle shown in Fig. 3 ($m = 2$) exists between the σ-axis and the curve $14056 \infty 1$ (the $6 \infty 1$ curve that passes through an infinitely distant point is shaded).

At the intersection of curve 14 (56) the cycle vanishes by merging with the separatrices of saddle points outside (inside) the cycle. At intersection with 405 the cycle vanishes owing to the formation on it of saddle-nodal points. At intersection of the shaded curve $6 \infty 1$ the cycle vanishes by merging with the unsteady cycle which comprises all singular points. That unstable cycle exists in region $6 \infty 1 \infty 36$; at intersection 63 it merges with the separatrices of saddle points,at intersection with $3 \infty 1$ it merges with separatrices of the $R = 0$ saddle point, and, then, splits in two unstable cycles (each of which covers only one $R \neq 0$) point). Such cycles exist in region $3 \infty 1 \infty 23$; at intersection 23 they merge with the separatrices of saddle points, and at intersection with $2 \infty 1$ they contract into focal points.

The case of $\sqrt{3} < \eta < 2,$ for $\sigma_3 > 0$ is shown in Figs. 2 and 4, where point 1 in the last figure is infinitely distant. A stable limit cycle exists between the σ - axis and curve $\infty 405678 \infty 9 \infty$ (curve 67 is shaded). At the intersection of curves $\infty 4$ and $\infty 9$ (56 and 78) the cycle vanishes by merging with the separatrices of saddle points outside (inside) the cycle. At intersection of 405 with $8 \infty 9$ saddle-nodal points are formed in the cycle. At intersection with the shaded curve 67 it merges with the unstable cycle which covers all singular points. That cycle exists in region 67; at intersection with the dash-line curve it merges with the separatrices of saddle points. In region 23 there exist between the solid and the upper dash-line curve three unstable cycles (each of which covers only one $R \neq 0$ point). At the intersection with the dash-line curve 23 they merge with the separatrices of saddle points and at intersection with the solid curve 23 they contract into focal points.

At transition of η through the value 2 points $7, 8$ and 3 pass to region $\sigma < \sigma_9$; and at its transition through the value $\sqrt{3}$ region 5678325 vanishes by contracting at point $\sigma = \sqrt{3}$.

The ends of dash-line curves in Figs. 3 and 4 correspond in Figs. 1 and 2 to saddle points that lie on straight lines 12. It can be shown [2, 4] that with increased η the double cycle splits in two, and that the shaded curves lie outside region $\sigma_1 < \sigma < 0.$ The results shown in [3] obtain for $\eta \to 0$ with $|\sigma_n| \to \infty.$

It was assumed above that $\gamma_2 < 0$ and $\eta \geqslant 0.$ The case of $\gamma_2 < 0$

and $\eta \leqslant 0$ does not need explanation. The case of $\gamma_2 = 0$ is considered below. The case $\gamma_2 > 0$ differs from that considered above in the direction of the trajectories of Eq. (2.1).

4. Let us consider the cases of violation of some of the introduced constraints.

It was assumed above that γ_2 is not small. If this is not so, the steady state solution Q (determined as before by (2.1) and (2.2)) is stable when $\gamma_0 < \gamma_* < 0$ where $\gamma_* \sim \varepsilon^2$ is determined by the nonsynchronous effect.

At considerable real frequency differences Eq. (1.4), although valid, is ineffective, since it requires additional transformations. Let, for example, the resonance with number $\zeta = n / m \neq 1$ be considered on the basis of the equation

$$dQ_1 / dt = Q_1 (p_0 - i\omega) + \varepsilon p_1 + \ldots$$

with $\zeta = 1$. After the substitution $Q_1 = Q_0 + Q_2$ where Q_0 is the steady state solution of that equation, the quantity Q_2 is sought in the form

$$Q_2 = Q \exp i\nu + f (Q, \varepsilon, \nu)$$

where f is a function periodic with respect to $\nu = (\zeta - 1) \omega t$ and $Q (t)$ satisfies Eq. (1.4) with the number $\zeta = n / m$.

Expansions (1.3) and (1.4) are valid for problems whose equations and boundary conditions contain high order derivatives with respect to t and which contain nonautonomous terms dependent on X that are nonlinear with respect to ε and X, and nonharmonic with respect to explicit t (see Sect. 5) below). The nonautonomous terms independent of X may be absent (related resonances are called parametric).

The case in which nonautonomous terms contain only harmonics of amplitudes $\sim \varepsilon$ and frequencies ω_n such that $| \omega_i - \omega_k | \gg | \gamma_0 |$ for $i \neq k$ does not present difficulties. The condition of resonances is defined by $m\Omega_0 \approx n_i\omega_i + \ldots + n_k\omega_k$; and the effective number is $n = | n_i | + \ldots + | n_k |$.

It was assumed above that problem (1.2) has only one critical value p_0. Let p_4 be another such value (γ_4 is small), then we have internal resonance, i.e., there exist relatively prime numbers m_0 and n_0 for which $p_4 \approx (n_0 / m_0) p_0$ and numbers m_4 and n_4 for which $p_4 \approx (n_4 / m_4) \omega$. The simultaneous consideration of equations for Q and Q_4 shows that approximation (1.4) is valid for Q if $m_0 + n_0 > 4$, $\chi < \chi_4$ and $\gamma_4 < \gamma_* < 0$, where $\gamma_* \sim Q^2 \sim \varepsilon^{2\chi}$; and, if the perturbation is almost periodic, the effective numbers n are to be used.

Note that all of the above is applicable to systems defined by ordinary differential equations.

5. As an example, let us consider the problem for $X = (\xi, w)$

$$X^{\cdot} + BX' + T = 0, \quad B = \begin{pmatrix} 01 \\ 10 \end{pmatrix}, \quad T = (0, \lambda w + \Phi') \tag{5.1}$$

$$\Phi = w^2 / (1 + \xi) - \xi + [(1 + \xi)^\beta - 1] / \beta = w^2 + h\xi^2 + \ldots,$$

$$(h = {}^1/_2 (\beta - 1)$$

$$(d\xi^{\cdot} + w)_0 = 0, \quad (w_1) = (1 + \xi)_1 2\varepsilon \cos \omega t$$

$$0 \leqslant x \leqslant l = 1 + 2 (\varepsilon / \omega) \sin \omega t, \quad a > 0, \quad \lambda \geqslant 0$$

where the dot and the primed denote differentiation with respect to t and x respectively, and the subscripts 0 and 1 at parentheses relate to $x = 0$, and l respectively. Problem (5. 1) defines the oscillation of gas induced by a rigid piston in a pipe with a chamber [2].

The substitution of coordinate $y = x / l$ yields the problem

$$X^{\cdot} - (l^{\cdot} / l)\, yX' + BX' / l + T_+ = 0, \quad T_+ = (0,\ \Phi' / l + \lambda w) \tag{5.2}$$

The subscripts 0 and 1 now relate to $y = 0$ and 1.

Solution of the linear autonomous problem is defined by the relationships

$$\begin{gathered} p_0 = ik, \quad k = {}^1\!/_2 i\lambda + (\varkappa^2 - {}^1\!/_4\, \lambda^2)^{1/2}, \quad \mathrm{tg}\, \varkappa = -a\varkappa \\ X_1 = (\cos\theta,\ -i\sin\theta), \quad \theta = \varkappa\,(y - 1) \end{gathered} \tag{5.3}$$

where X_1 is taken for $\lambda = 0$; and in this approximation the eigenfunction of the conjugate problem is $Z = X_1$.

Below the quantity ${}^1\!/_2\,\omega$ is assumed to be close to the first root $\varkappa = \varkappa_1$ of Eq. (5.3), and a is selected so that internal resonances do not violate approximation (1.4) when $\zeta = {}^1\!/_2$; such values of a do exist [2].

In (1.3) we have the term $\bar{Q}\varepsilon E^{1/2} Y\,(x)$; and coefficient p_1 is determined by the problem for Y (it is sufficient to determine it for $\lambda = 0$ and $\omega = 2\varkappa$). First, it is necessary to determine X_2 using the problem

$$2\varkappa i X_2 + BX_2' = 0, \quad (w)_1 = 1, \quad (2i\varkappa\alpha\xi + w)_0 = 0$$

whose solution is

$$\begin{gathered} \xi_2 = -i\,(\sin 2\theta + b\cos 2\theta), \quad w_2 = \cos 2\theta - b\sin 2\theta \\ b = (1 + 3a^2\varkappa^2) / (2a^3\varkappa^3) \end{gathered} \tag{5.4}$$

The problem for Y is of the form

$$\begin{gathered} i\varkappa Y + BY + S = 0, \quad (w)_1 = 1, \quad (i\varkappa a\xi + ap_1\cos\varkappa + w)_0 = 0 \\ S = p_1 X_1 - y\bar{X}_1' - B\bar{X}_1' / (2i\varkappa) + (0,\ \Phi_0') \\ \Phi = 2\,(\bar{w}_1 w_2 + h\,\bar{\xi}_1\xi_2) \end{gathered} \tag{5.5}$$

The substitution

$$Y = Y_* + A / (i\varkappa), \quad A = (-1 / a - ip_1\cos\varkappa,\ i\varkappa)$$

reduces (5.5) to the problem for Y_* with homogeneous conditions and the free term $\Psi = A + S$. Solution of that problem exists when $\langle \Psi \cdot Z \rangle = 0$; from this we obtain

$$(1 + a\cos^2\varkappa)\, p_1 = -i\int_0^1 \Phi_0' \sin\theta\, dy - \frac{1}{2} \tag{5.6}$$

Equations (5.3)-(5.6) show that p_1 is real, and if in problem (5.1) $l = 1$ the right-hand side of (5.6) is decreased by half.

The coefficient p_2 in (1.4) for problem (5.1) proves to be imaginary [2]. In accordance with Sect. 4 the steady state solution Q is determined by (1.4) when the quantities $p_{1,2}$ are not small, and the solution is stable when $\lambda > \lambda_* \sim \varepsilon^2$.

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